Lecture 5

Fourier Transform

f(t) – Function; F(ω) – Fourier harmonic

$$F(\omega)=\int_{-\infty}^{+\infty} f(t)\exp(\mathrm{i}\omega t)\mathrm{d}t$$

$$f(t)=\frac{1}{2\pi}\int_{-\infty}^{+\infty} F(\omega)\exp(-\mathrm{i}\omega t)\mathrm{d}\omega$$

F = F(ω) : Spectral Distribution, Spectrum

Fourier Transform

Time / Frequency

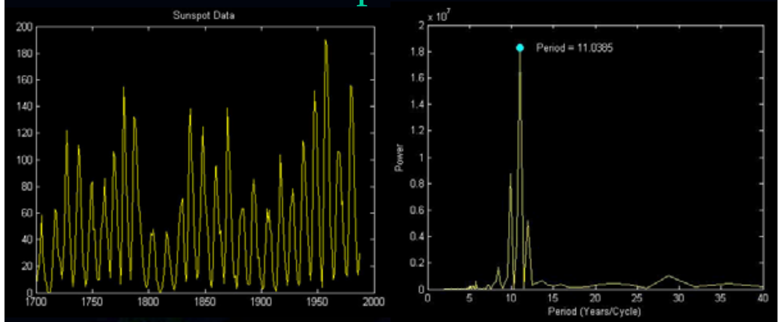
$$F(\omega)=\int_{-\infty}^{+\infty} f(t)\exp(\mathrm{i}\omega t)\mathrm{d}t$$

Co-ordinate / Wave-number

$$F(k)=\int_{-\infty}^{+\infty} f(x)\exp(\mathrm{i}kx)\mathrm{d}x$$

Temporal (Spatial) spectrum: **Spectral Analysis**

Sunspot Data



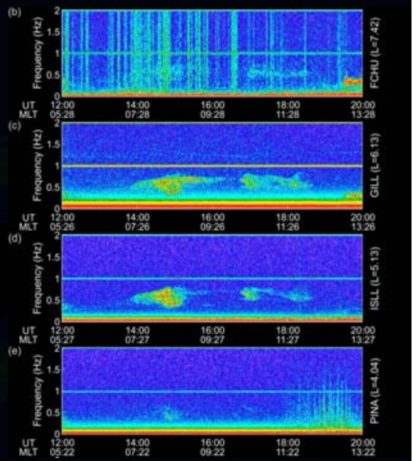
Time series Spectrum

```
load sunspot.dat
Year = sunspot(:,1); Wolf = sunspot(:,2);
```

Spectrogram

Time evolution
of the temporal
spectrum (frequencies)

Solar wind pressure
and magnetic field
Fourier spectrograms
(CARISMA)



FT: Properties

Linear Superposition

$$f_1(t) + f_2(t) \to F_1(\omega) + F_2(\omega)$$

Correlation

$$\int_{-\infty}^{+\infty} f_1(\tau)\, f_2(t-\tau)\mathrm{d}\tau \to F_1(\omega) F_2(\omega)$$

Convolution

$$\int_{-\infty}^{+\infty} f_1(\tau)\, f_2(t+\tau)\mathrm{d}\tau \to F_1(\omega) F_2^{\,*}(\omega)$$

Discrete Fourier Transform

Continuous: ***X= (-∞ … ∞)***

Discrete: ***X_k = (X_1 … X_N)***

$$X_k = \sum_{n=0}^{N-1} x_n \exp\left(-i2\pi \frac{k}{N} n\right)$$

$$x_n = \frac{1}{N} \sum_{k=0}^{N-1} X_k \exp\left(i2\pi \frac{k}{N} n\right)$$

k = 0 … (N-1) **Discrete Spectrum**

DFT constraints

Length-Scales:

Domain Size: ***$L = X_N - X_1$***

Step Size: ***$\Delta X = X_k - X_{k-1}$, $\Delta X = L/(N-1)$***

Wave-Numbers:

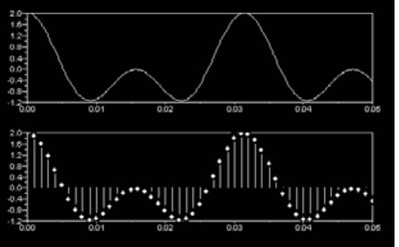
Maximal wave-number: ***$K = 2\pi / \Delta X$***

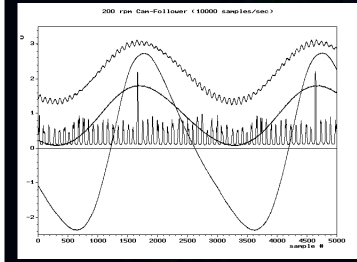
Minimal wave-number: ***$\Delta K = 2\pi / L$***

Number of Fourier Harmonics: **N**
(sampling rate, resolution)

Sampling

Discretization:
Sample continuous function



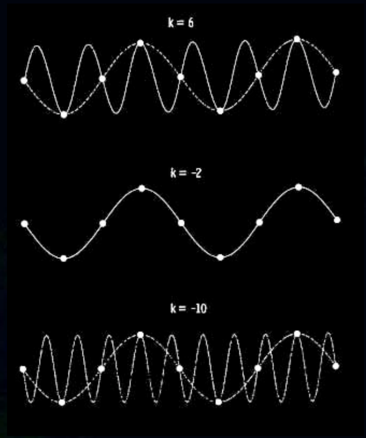


Nyquist critical frequency:

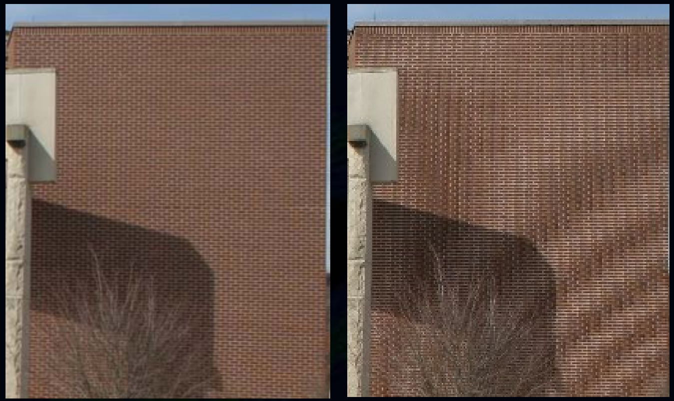
$\omega_c = 1 / 2\Delta$

$\boxed{\omega < \omega_c}$

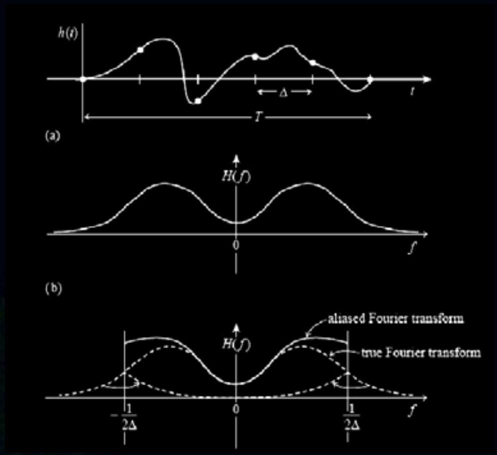
Sampling



Aliasing

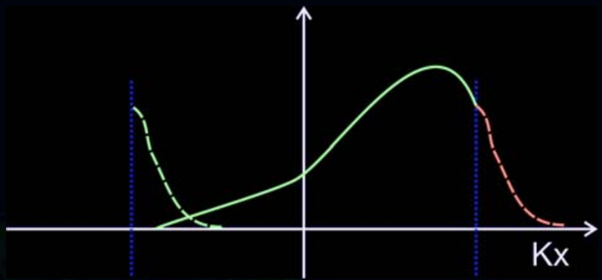


Aliasing



Aliasing

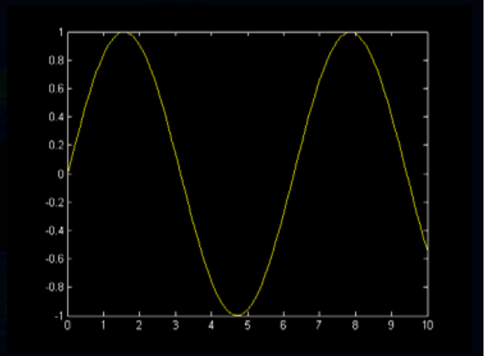
Aliasing defects to spectral power



$$k_x \rightarrow -k_x$$

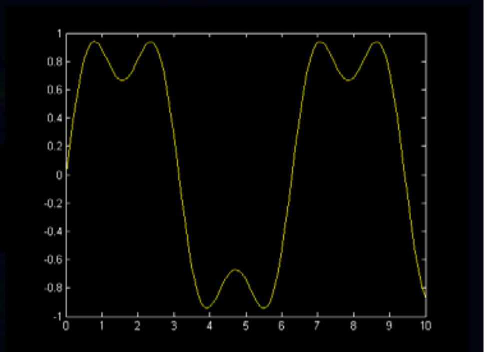
Number of Harmonics

```
t = (0:0.1:10);
y = sin(t);
plot(t,y);
```



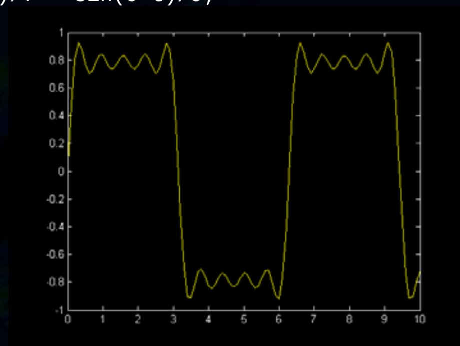
Number of Harmonics

```
t = (0:0.1:10);
y = sin(t) + sin(3*t)/3;
plot(t,y);
```

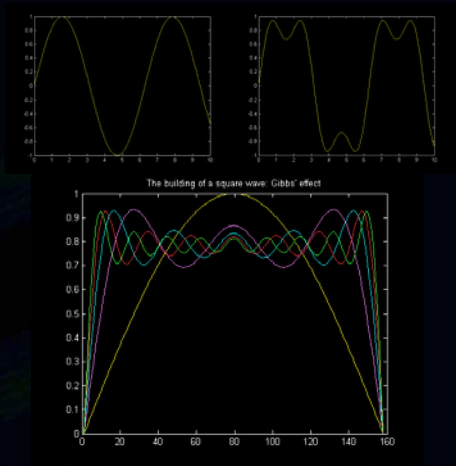


Number of Harmonics

```
t = (0:0.1:10);
y = sin(t) + sin(3*t)/3 + ...
sin(5*t)/5+ sin(7*t)/7 + sin(9*t)/9;
plot(t,y);
```



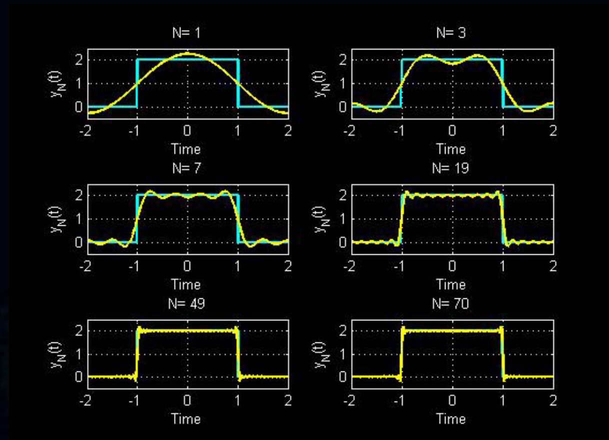
Gibbs Effect



N = 1…9

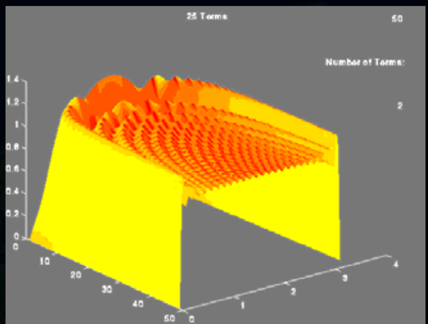
Density of the spectrum

Gibbs Effect



Gibbs in 2D

(X,Y) / (Kx, Ky)



Fast Fourier Transform

DFT:

***O*(N²)** calculation process

FFT Algorithm: ($N = 2^m$)

***O*(N log N)** calculation process

(Cooley, Turkey 1960, … Gauss 1805)

$N = 10^6$ FFT : 30sec

FT : 2 week

PDE: Spectral Method

PDE:
$$\frac{\partial}{\partial t}A(x,t)=c^2\frac{\partial^2}{\partial x^2}A(x,t)$$

Fourier decomposition in SPACE:
$$a(k,t)=\int_{-\infty}^{+\infty}A(x,t)\exp(ikx)\mathrm{d}x$$

$$\int\left\{\boxed{\frac{\mathrm{d}}{\mathrm{d}t}a(k,t)+c^2k^2a(k,t)}\right\}\exp(ikx)\mathrm{d}x=0$$

ODE solver: $a = a(k,t)$

Inverse Fourier transform: $A = A(x,t)$

Spectral Simulations

Discrteize PDE: Sampling

DFT: mostly FFT

1. Transform PDE to spectral ODE
2. Solve ODE (e.g., R-K)
3. Inverse transform to reconstruct solutions

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Spectral Method: Features

Initial Value Problem

- Calculate initial values in k-space

Boundary Value Problem

- Integrate boundaries into k-space

Spatial Inhomogeneities

- Introduce numerical variables to homogenize
- Integrate during reconstruction

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Spectral Method: Problems

1. **Shocks**

Discontinuity: $\Delta \to 0$

$K_{cr} = 1/2\Delta \to \infty$

$K_{max} < K_{cr}$

2. **Complex Boundaries**

Ill-known numerical instabilities;

3. **Nonlinearities**

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Spectral Method: Variants

- Galerkin Method (finite element methods)
- Tau Method
- Pseudo-spectral Method

Comparison

Spectral Method:
Linear combination of continuous functions;
Global approach;

Finite Difference, Flux conservation:
Array of piecewise functions;
Local approach;

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+ / -

+ (very) fast for smooth solutions
+ Exponential convergence
+ Best for turbulent spectrum

- Shocks
- Inhomogeneities
- Complex Boundaries
- Need for serial reconstruction (integration)

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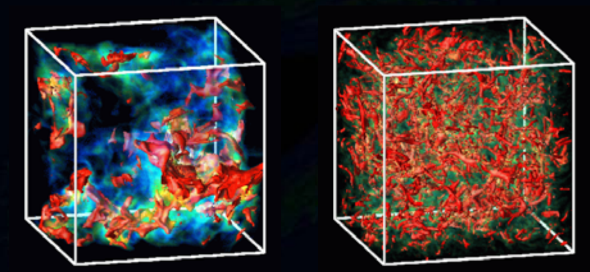
Chaotic flows

Post-processing:

Partial Reconstruction at different length-scales



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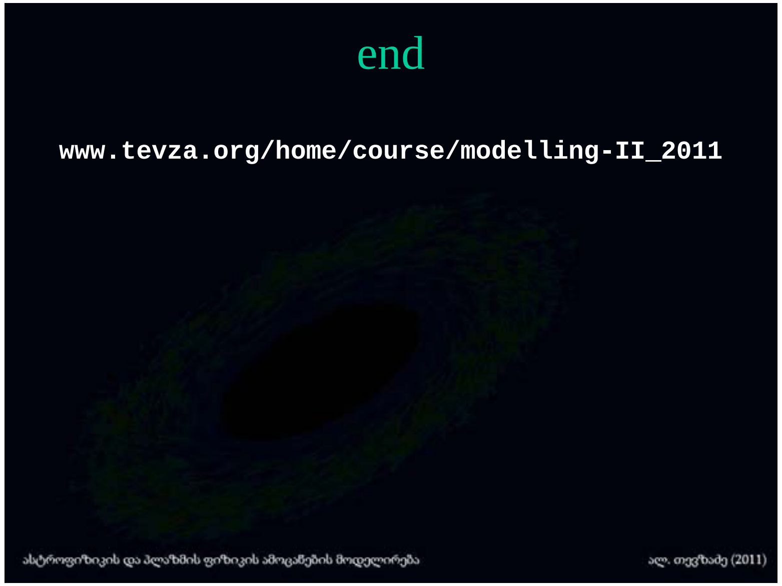
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