ლექცია 5

Fourier Transform

f(t) – Function; F(ω) – Fourier harmonic

$$F(\omega) = \int_{-\infty}^{+\infty} f(t)\exp(\mathrm{i}\omega t)\mathrm{d}t$$

$$f(t) = \frac{1}{2\pi}\int_{-\infty}^{+\infty} F(\omega)\exp(-\mathrm{i}\omega t)\mathrm{d}\omega$$

F = F(ω) : Spectral Distribution, Spectrum

Fourier Transform

Time / Frequency

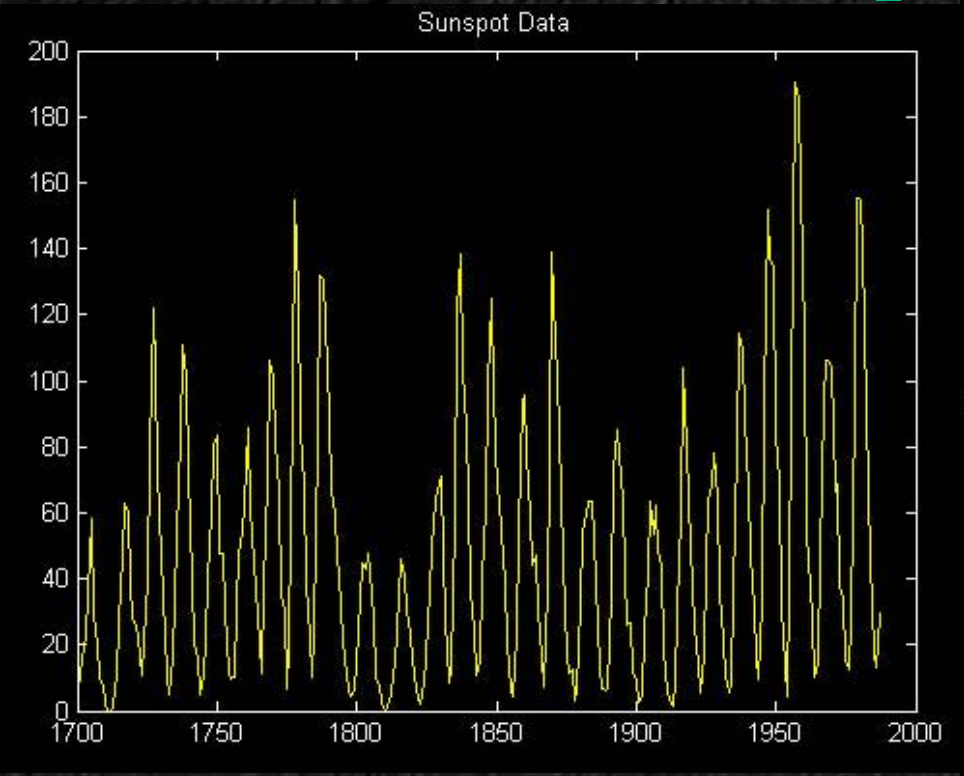
$$F(\omega) = \int_{-\infty}^{+\infty} f(t) \exp(\mathrm{i}\omega t) \mathrm{d}t$$

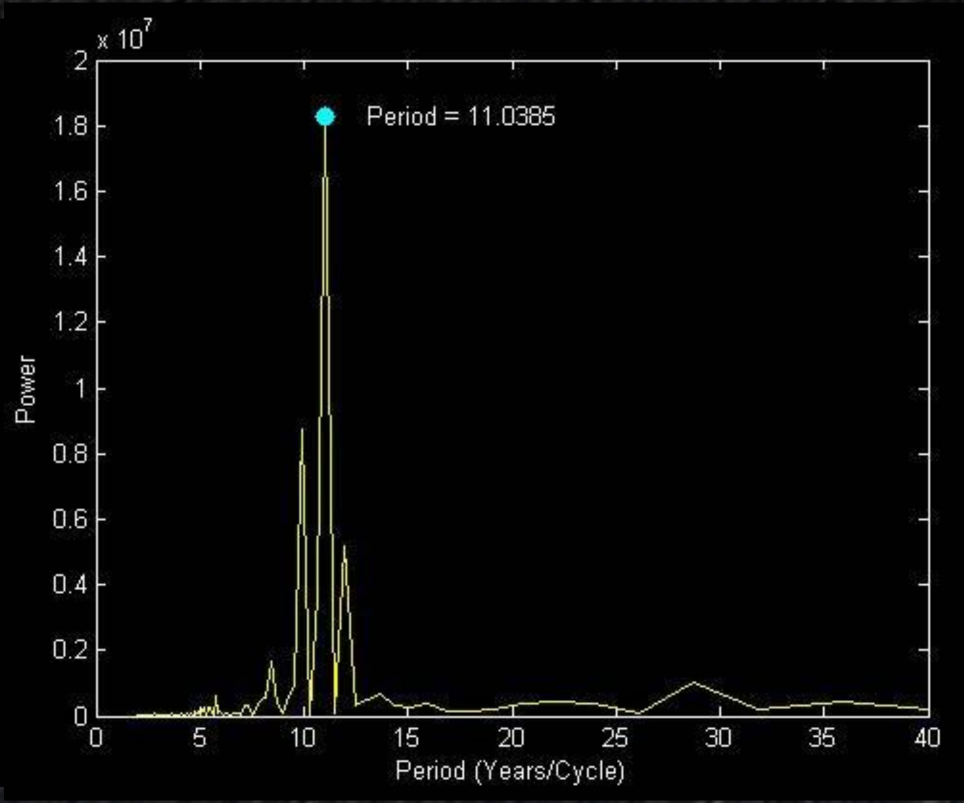
Co-ordinate / Wave-number

$$F(k) = \int_{-\infty}^{+\infty} f(x) \exp(\mathrm{i}kx) \mathrm{d}x$$

Temporal (Spatial) spectrum: **Spectral Analysis**

Sunspot Data





Time series

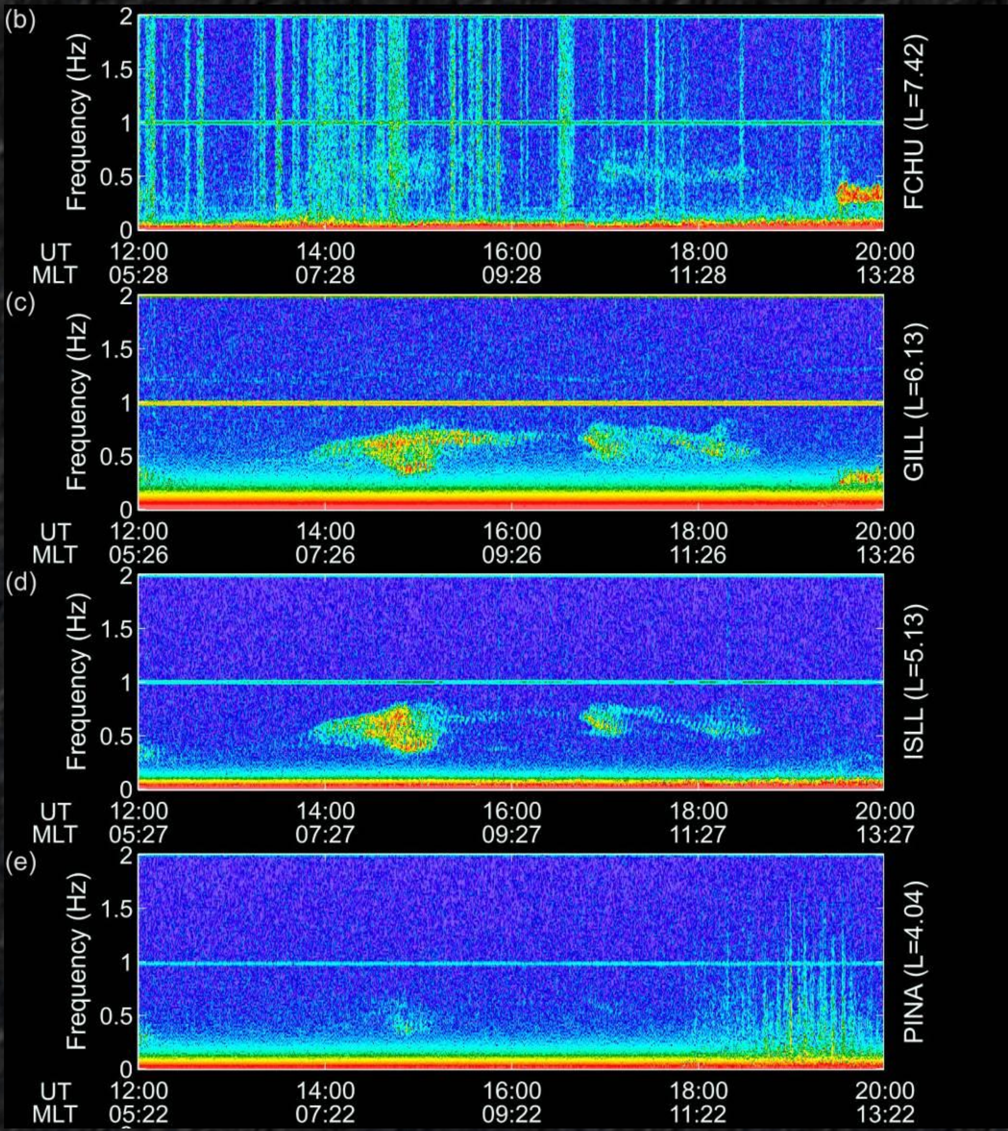
Spectrum

```
load sunspot.dat
Year = sunspot(:,1); Wolf = sunspot(:,2);
```

Spectrogram

Time evolution
of the temporal
spectrum (frequencies)

Solar wind pressure
and magnetic field
Fourier spectrograms
(CARISMA)



FT: Properties

Linear Superposition

$$f_1(t) + f_2(t) \to F_1(\omega) + F_2(\omega)$$

Correlation

$$\int_{-\infty}^{+\infty} f_1(\tau) f_2(t-\tau)\mathrm{d}\tau \to F_1(\omega)F_2(\omega)$$

Convolution

$$\int_{-\infty}^{+\infty} f_1(\tau) f_2(t+\tau)\mathrm{d}\tau \to F_1(\omega)F_2^{*}(\omega)$$

Discrete Fourier Transform

Continuous: $X = (-\infty \ldots \infty)$

Discrete: $X_k = (X_1 \ldots X_N)$

$$X_k = \sum_{n=0}^{N-1} x_n \exp\left(-i2\pi \frac{k}{N} n \right)$$

$$x_n = \frac{1}{N} \sum_{k=0}^{N-1} X_k \exp\left(i2\pi \frac{k}{N} n \right)$$

$k = 0 \ldots (N\text{-}1)$

Discrete Spectrum

DFT constraints

Length-Scales:

Domain Size: $L = X_N - X_1$

Step Size: $\Delta X = X_k - X_{k-1}$, $\Delta X = L/(N-1)$

Wave-Numbers:

Maximal wave-number: $K = 2\pi / \Delta X$

Minimal wave-number: $\Delta K = 2\pi / L$

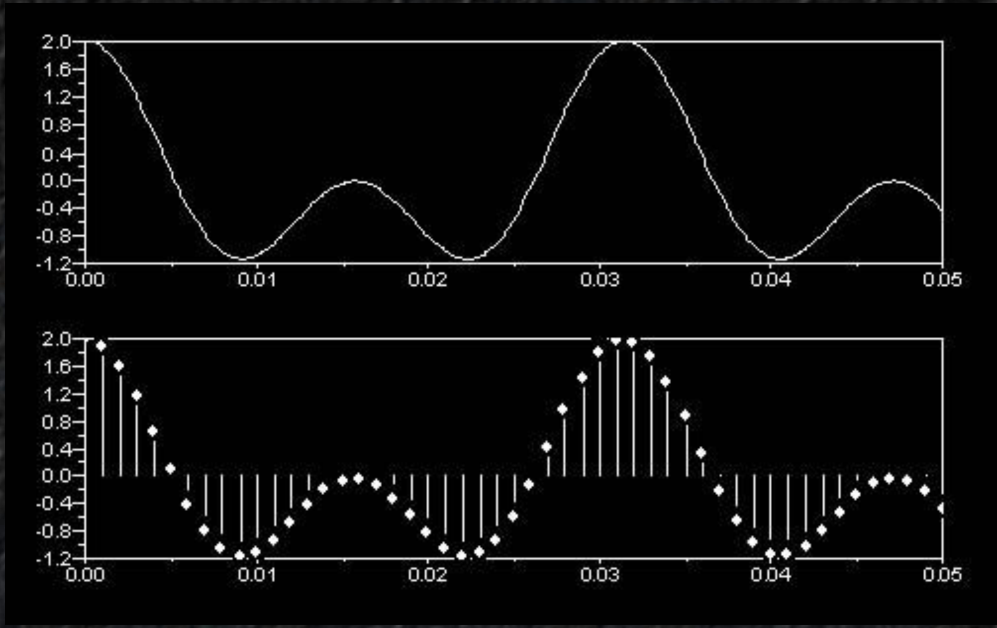
Number of Fourier Harmonics: **N**

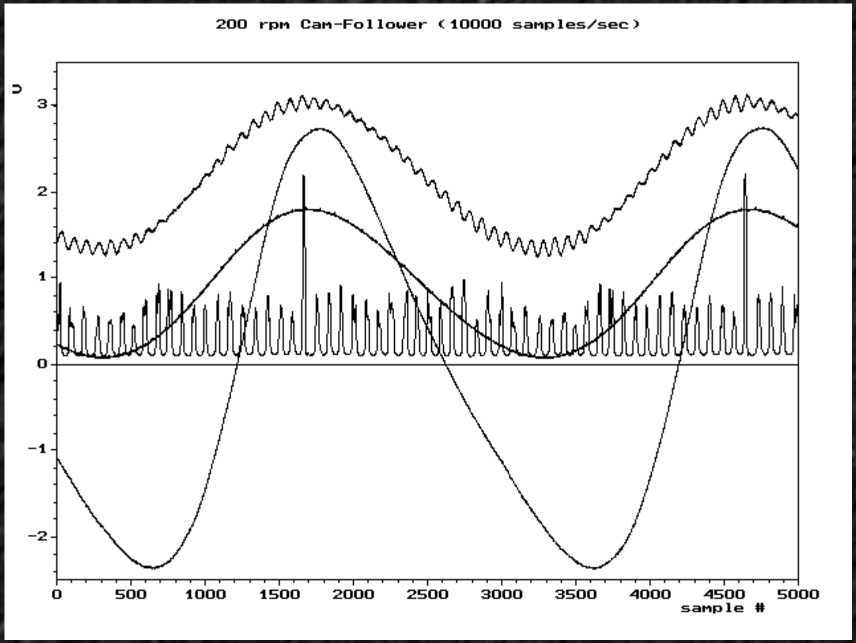
(sampling rate, resolution)

Sampling

Discretization:

Sample continuous function



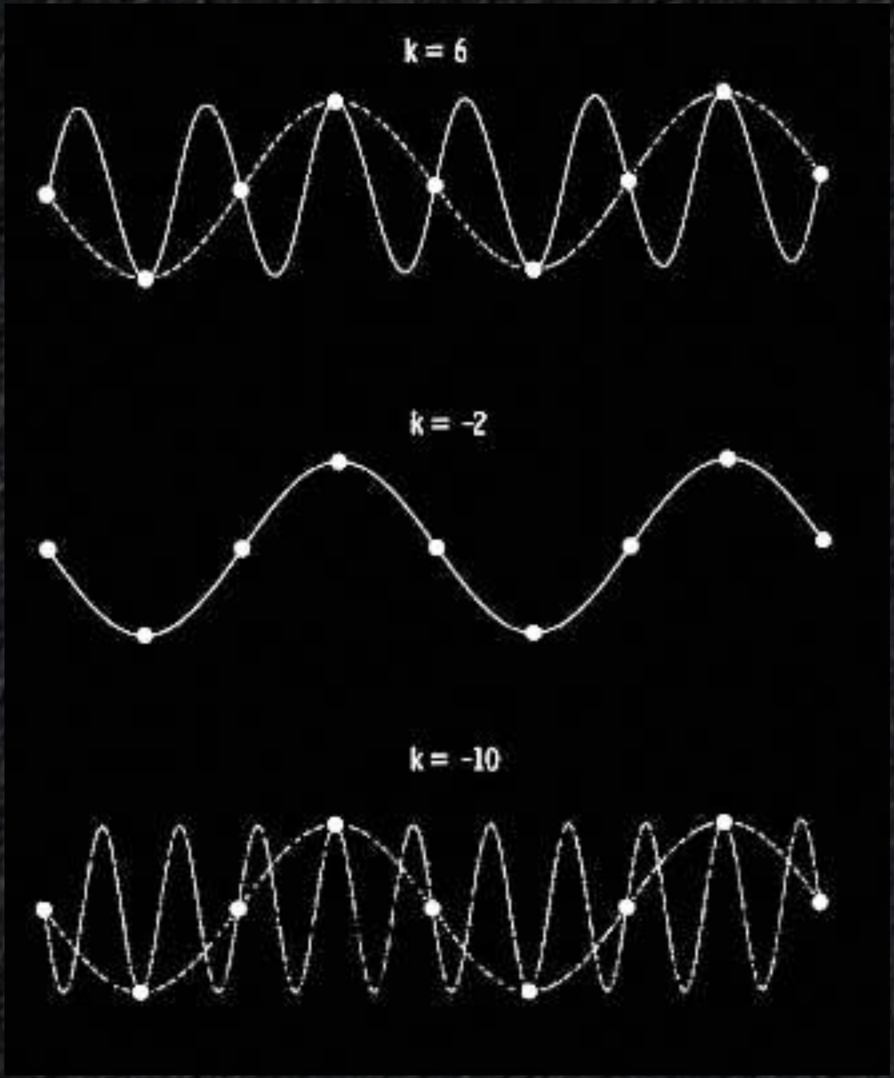


Nyquist critical frequency:

$\omega_c = 1 / 2\Delta$

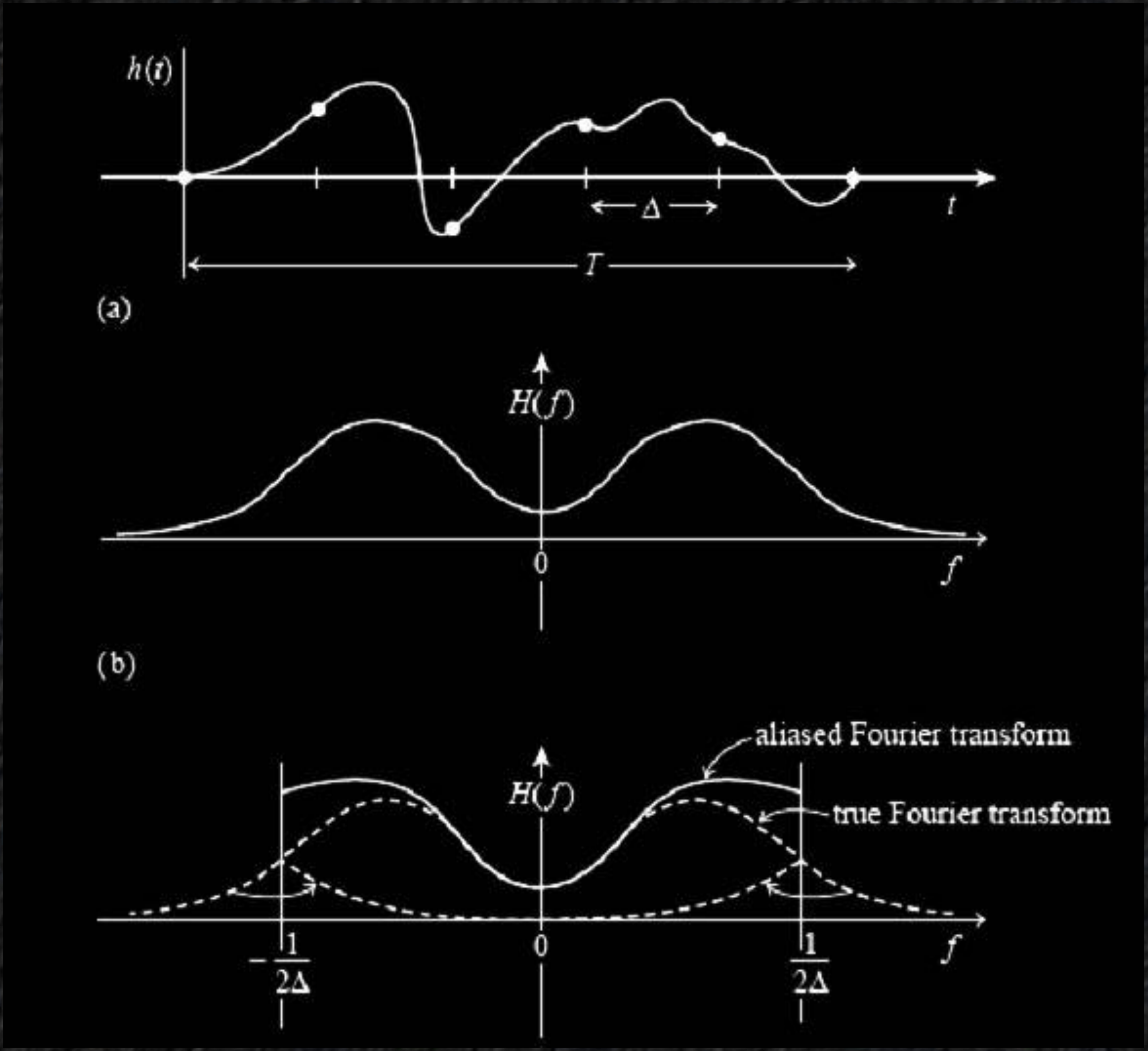
$\omega < \omega_c$

Sampling



Aliasing

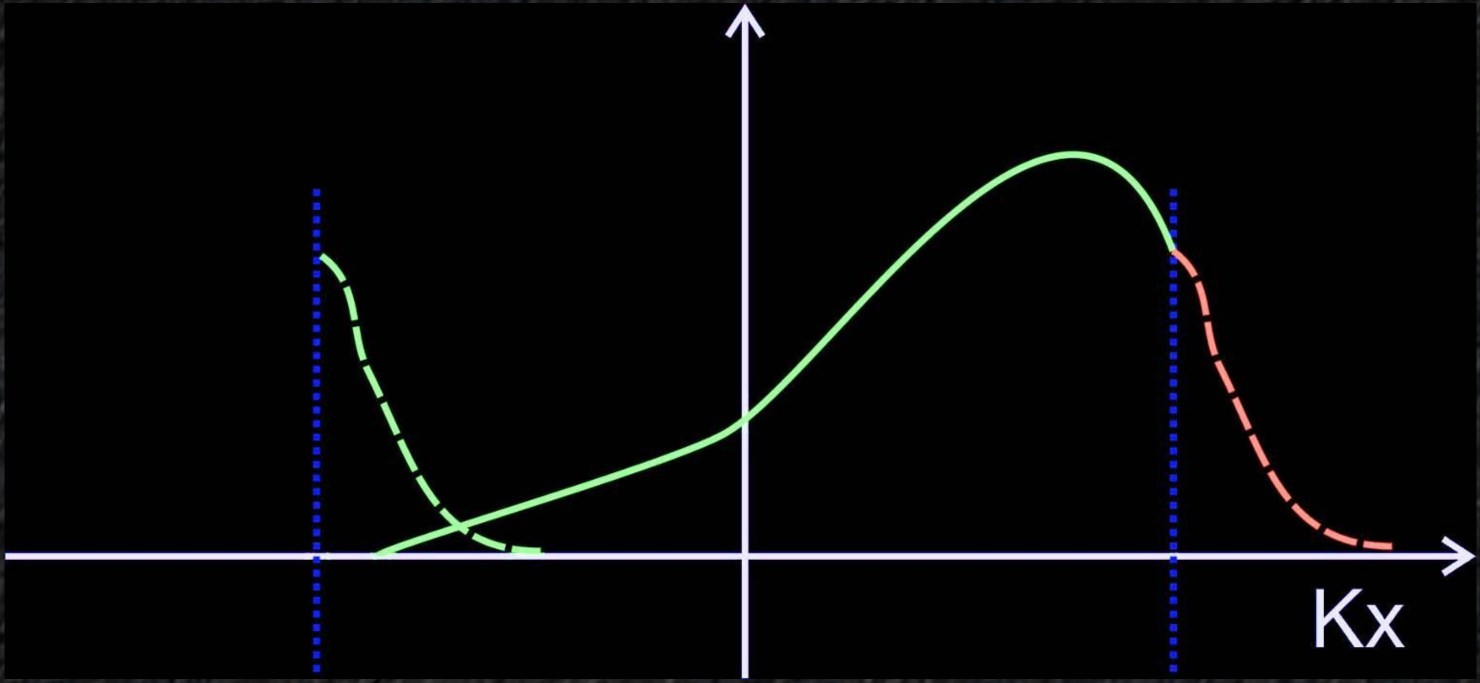
Aliasing



h(t)
Δ
t
T
(a)
H(f)
0
f
(b)
aliased Fourier transform
true Fourier transform
H(f)
0
f
1
2Δ
1
2Δ

Aliasing

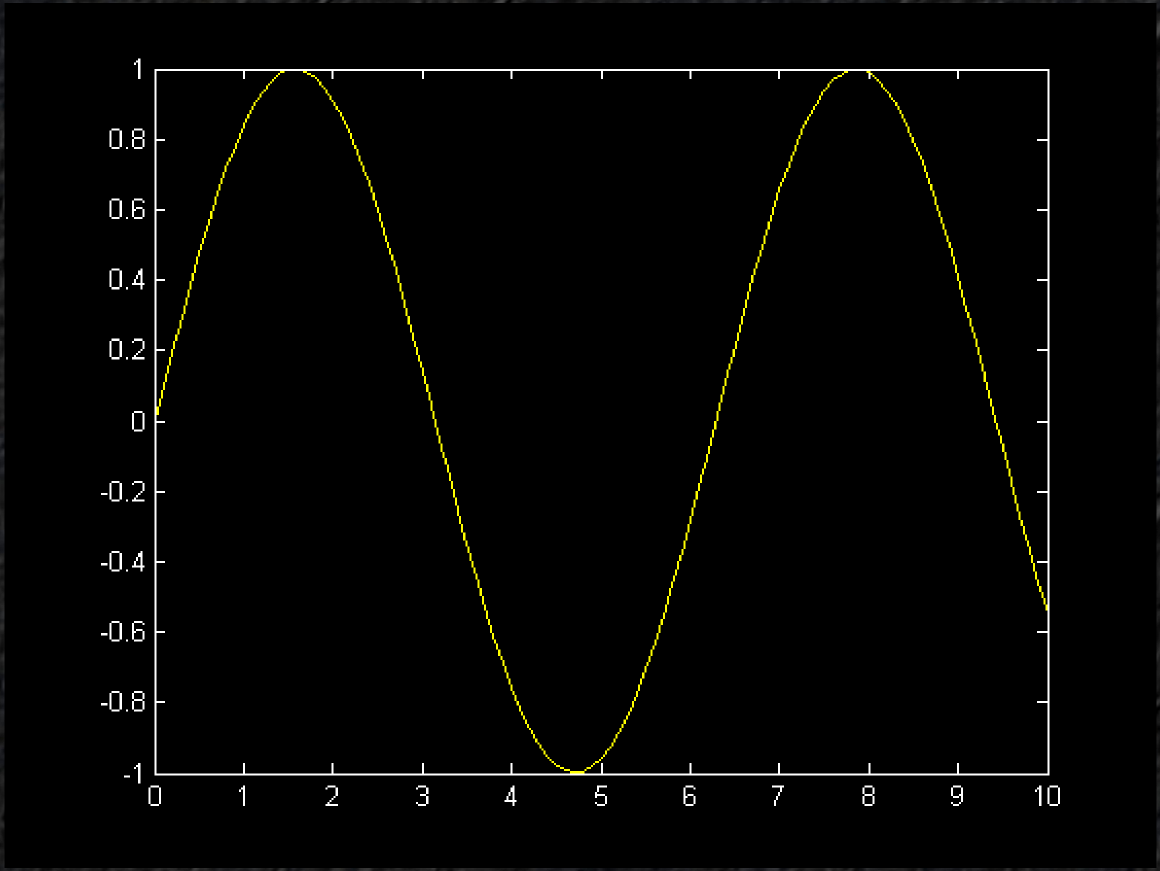
Aliasing defects to spectral power



$$k_x \rightarrow -k_x$$

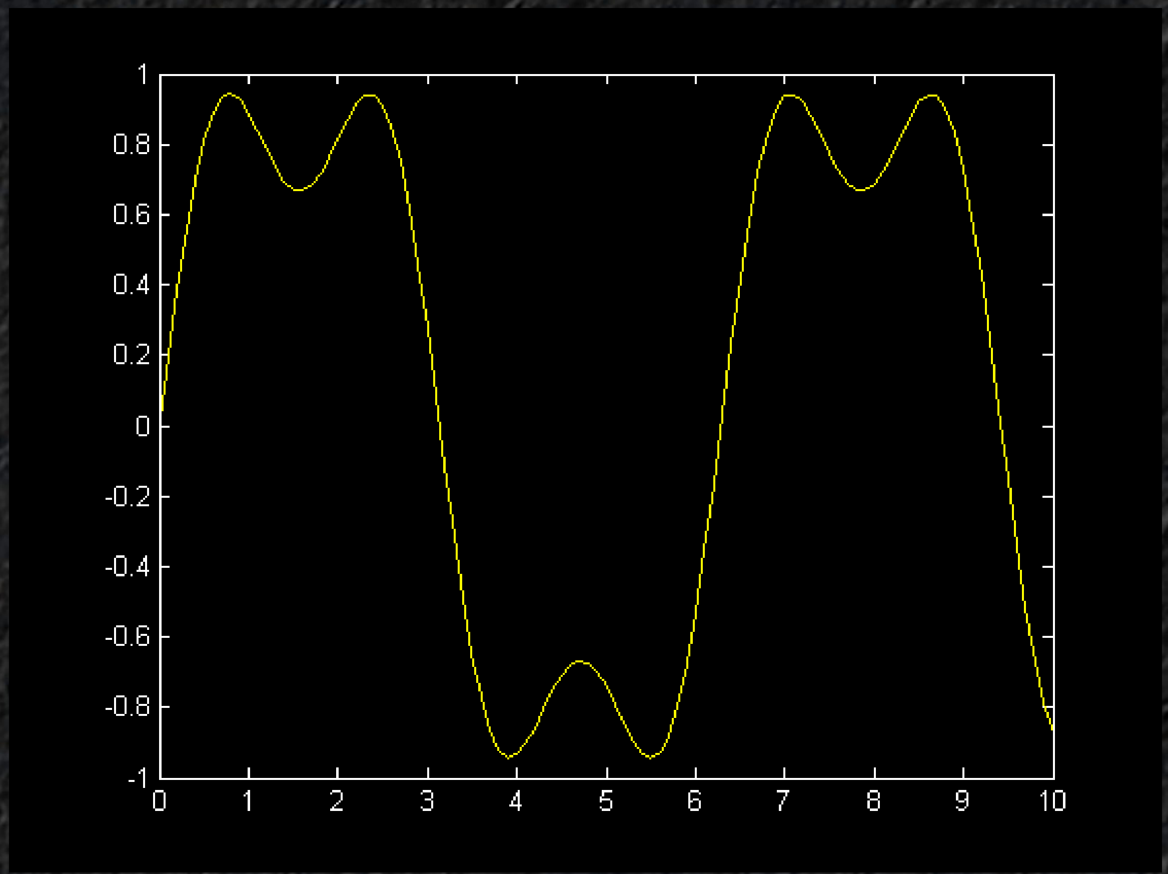
Number of Harmonics

```
t = (0:0.1:10);
y = sin(t);
plot(t,y);
```



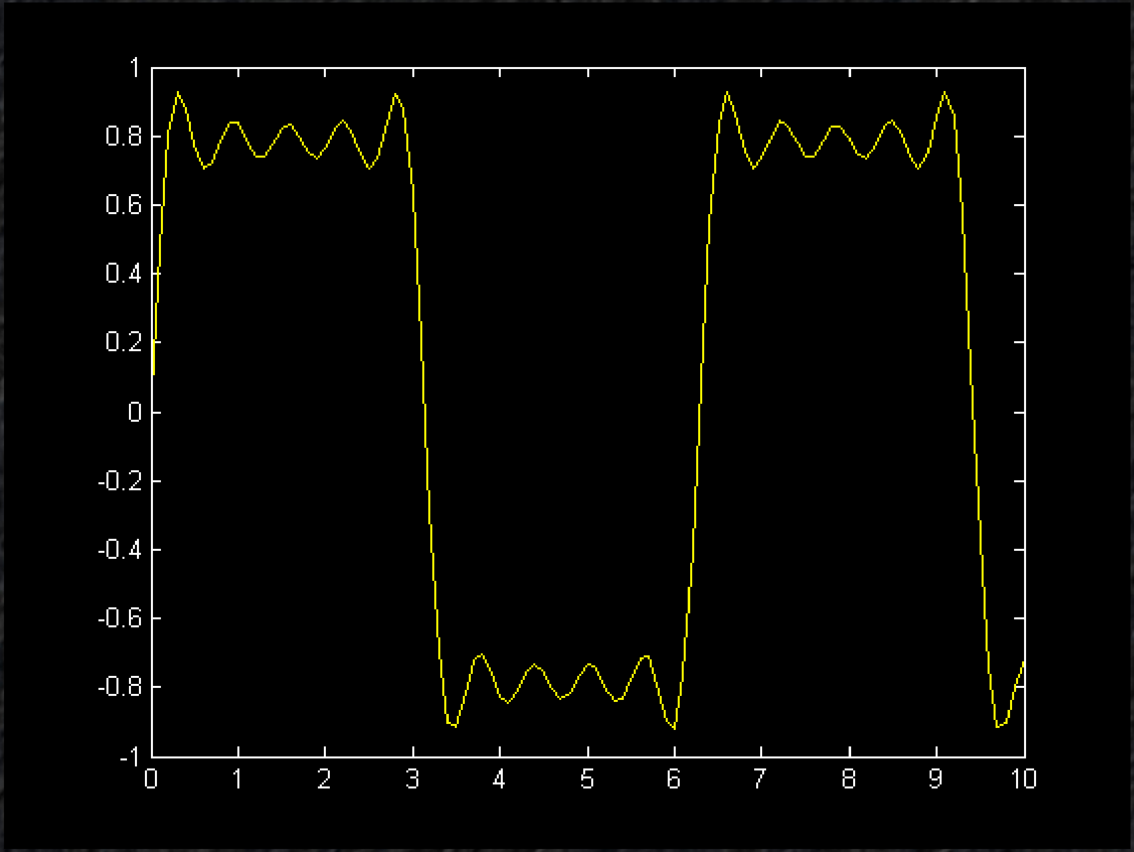
Number of Harmonics

```
t = (0:0.1:10);
y = sin(t) + sin(3*t)/3;
plot(t,y);
```

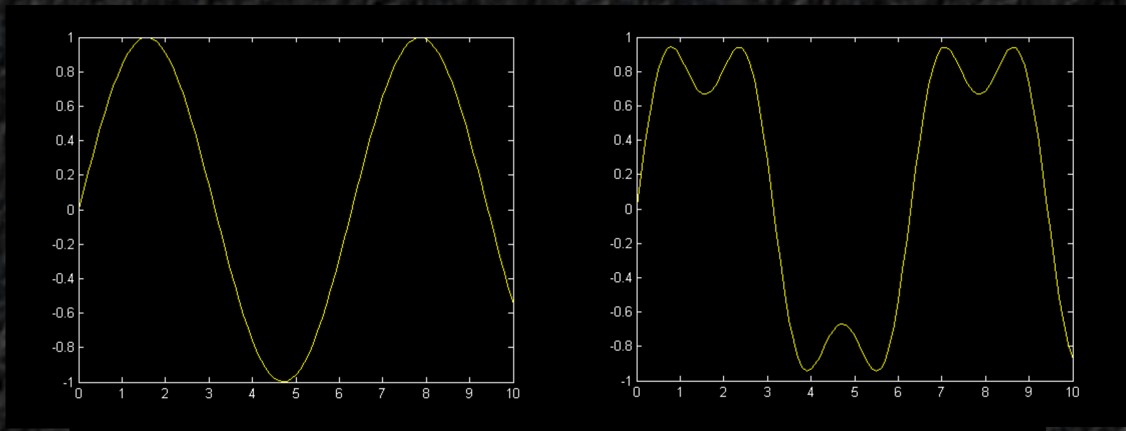


Number of Harmonics

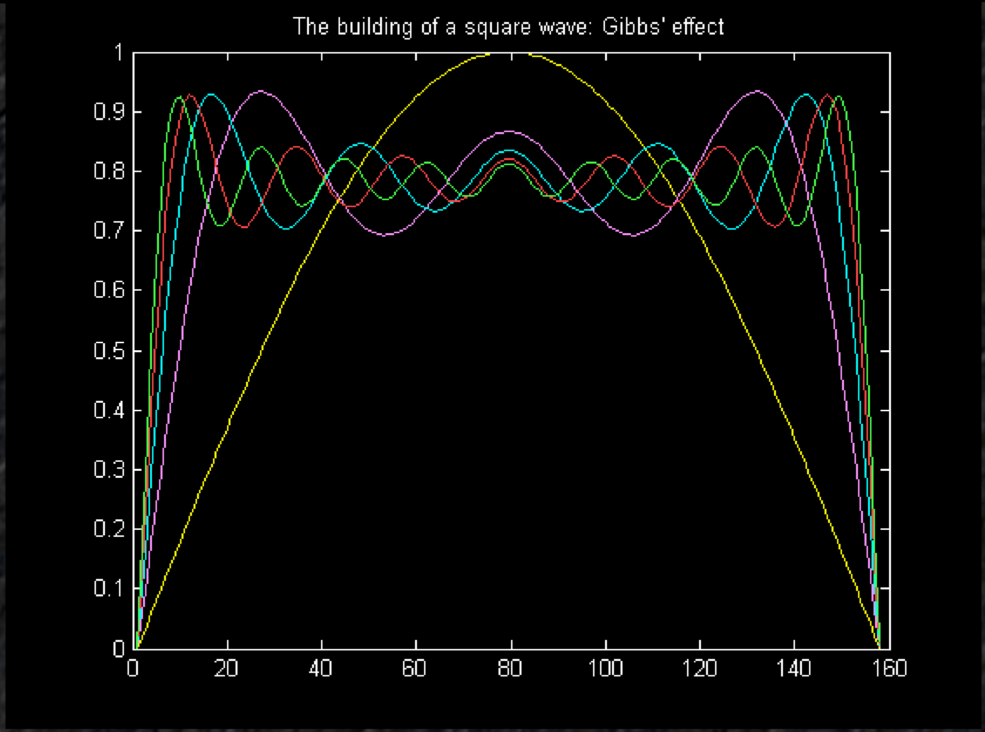
```
t = (0:0.1:10);
y = sin(t) + sin(3*t)/3 + ...
sin(5*t)/5+ sin(7*t)/7 + sin(9*t)/9;
plot(t,y);
```



Gibbs Effect

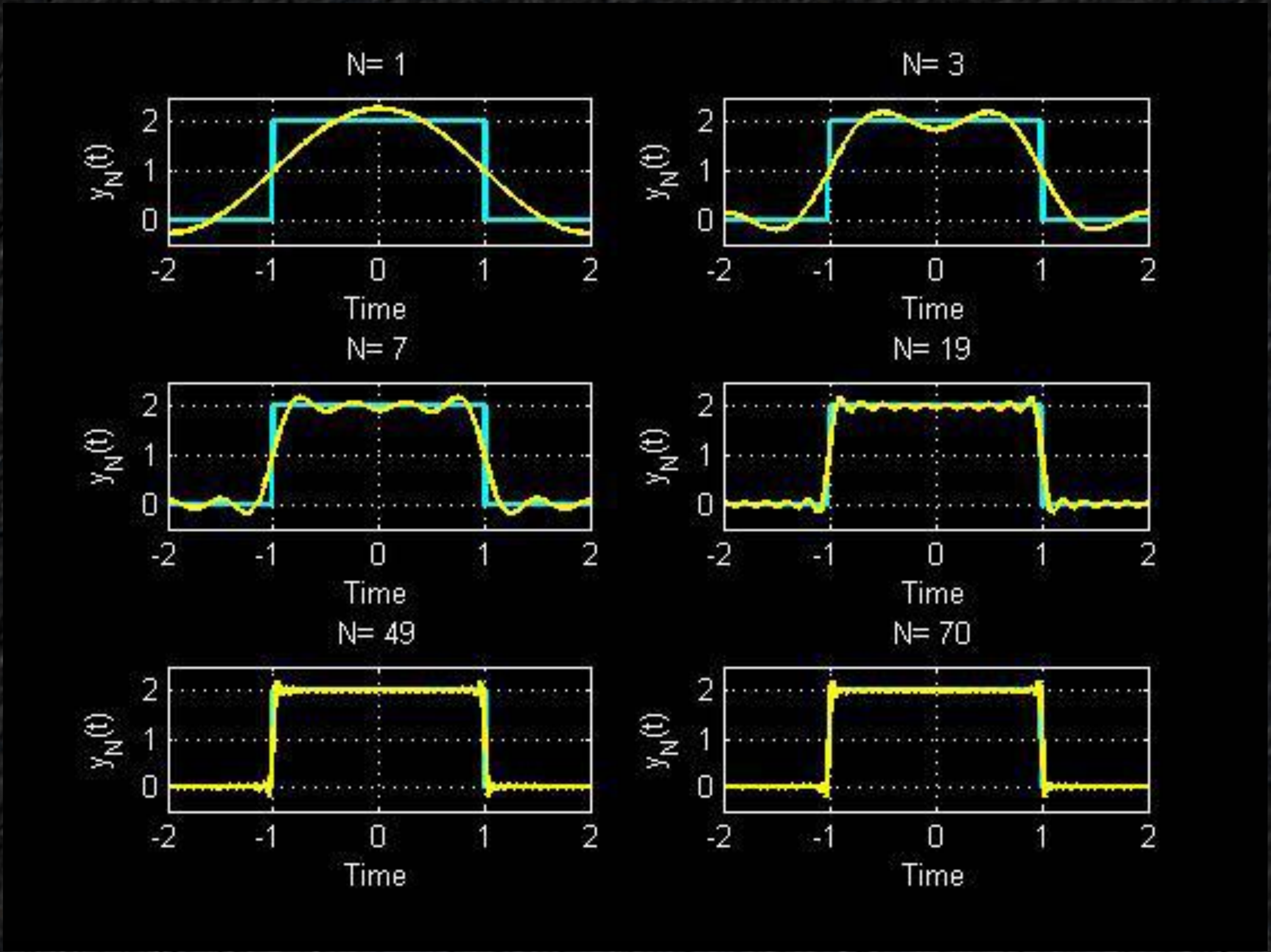


N = 1…9



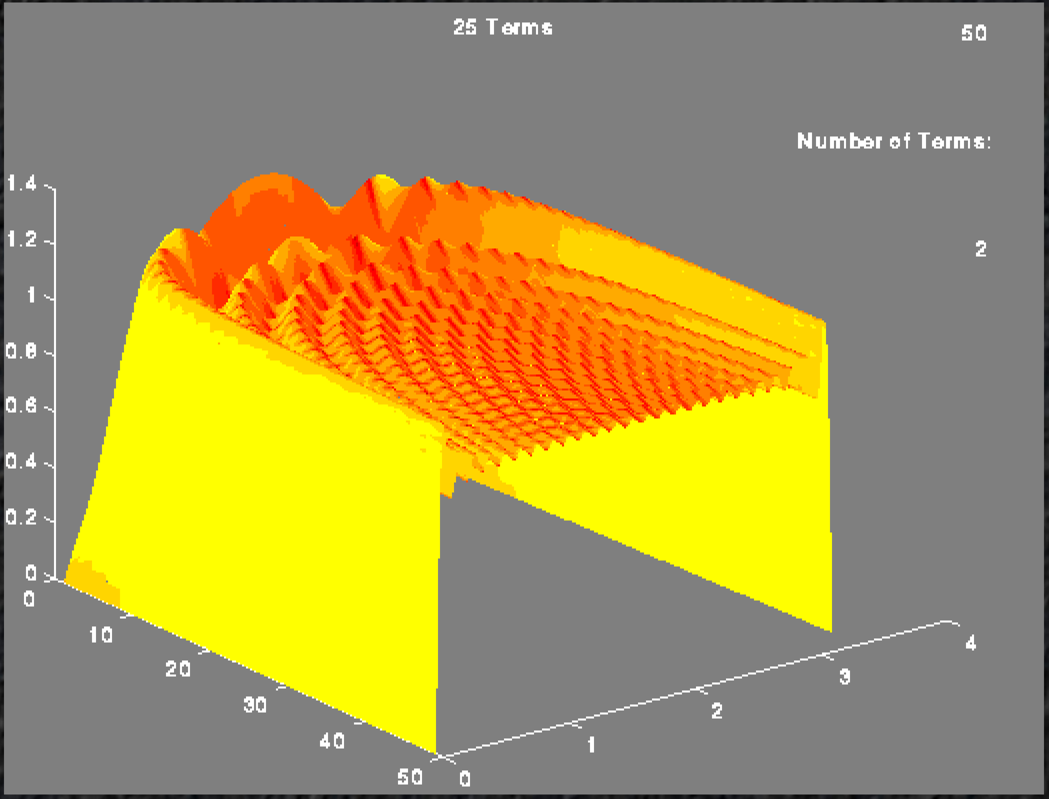
Density of the spectrum

Gibbs Effect



Gibbs in 2D

(X,Y) / (Kx, Ky)



Fast Fourier Transform

DFT:

***O*(N^2)** calculation process

FFT Algorithm: ($N = 2^m$)

***O*(N log N)** calculation process

(Cooley, Turkey 1960, ... Gauss 1805)

$N = 10^6$ FFT : 30sec

FT : 2 week

PDE: Spectral Method

PDE:

$$\frac{\partial}{\partial t}A(x,t)=c^2\frac{\partial^2}{\partial x^2}A(x,t)$$

Fourier decomposition in SPACE:

$$a(k,t)=\int_{-\infty}^{+\infty}A(x,t)\exp(\mathrm{i}kx)\mathrm{d}x$$

$$\int\left\{\frac{\mathrm{d}}{\mathrm{d}t}a(k,t)+c^2k^2a(k,t)\right\}\exp(ikx)\mathrm{d}x=0$$

ODE solver: $a = a(k,t)$

Inverse Fourier transform: $A = A(x,t)$

Spectral Simulations

Discrteize PDE: Sampling

DFT: mostly FFT

1. Transform PDE to spectral ODE
2. Solve ODE (e.g., R-K)
3. Inverse transform to reconstruct solutions

Spectral Method: Features

<u>Initial Value Problem</u>

- Calculate initial values in k-space

<u>Boundary Value Problem</u>

- Integrate boundaries into k-space

<u>Spatial Inhomogeneities</u>

- Introduce numerical variables to homogenize
- Integrate during reconstruction

Spectral Method: Problems

1. **Shocks**

Discontinuity: $\Delta \rightarrow 0$

$$K_{cr} = 1/\,2\Delta \;\rightarrow \infty$$

$$K_{max} < K_{cr}$$

2. **Complex Boundaries**

Ill-known numerical instabilities;

3. **Nonlinearities**

Spectral Method: Variants

- Pseudo-spectral Method

Pseudo-spectral basis:

Legendre polynomials;

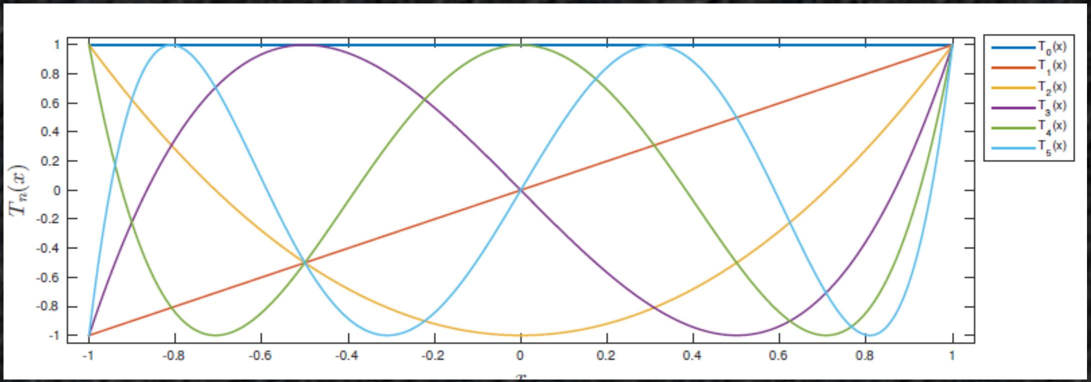
Chebishev polynomials;

Expansion coefficients: colocation, Galerkin,…

Chebishev polynomials

$$\langle T_n, T_m \rangle \equiv \int_{-1}^{1} \frac{T_n(x) \cdot T_m(x)}{\sqrt{1 - x^2}}\, dx = \begin{cases} 0, & m \neq n\,, \\ \pi, & m = n = 0\,, \\ \frac{\pi}{2}, & m = n > 0\,. \end{cases}$$

$$\begin{aligned} T_0(x) &= 1\,, \\ T_1(x) &= x\,, \\ T_2(x) &= 2\,x^2 - 1\,, \\ T_3(x) &= 4\,x^3 - 3\,x\,, \\ T_4(x) &= 8\,x^4 - 8\,x^2 + 1\,, \\ T_5(x) &= 16\,x^5 - 20\,x^3 + 5\,x\,. \end{aligned}$$



$$T_{n+1}(x) = 2\,x\,T_n(x) - T_{n-1}(x)\,.$$

Example

$$u_t = \nu u_{xx} \, ,$$
$$u(x, 0) = u_0(x) \, .$$

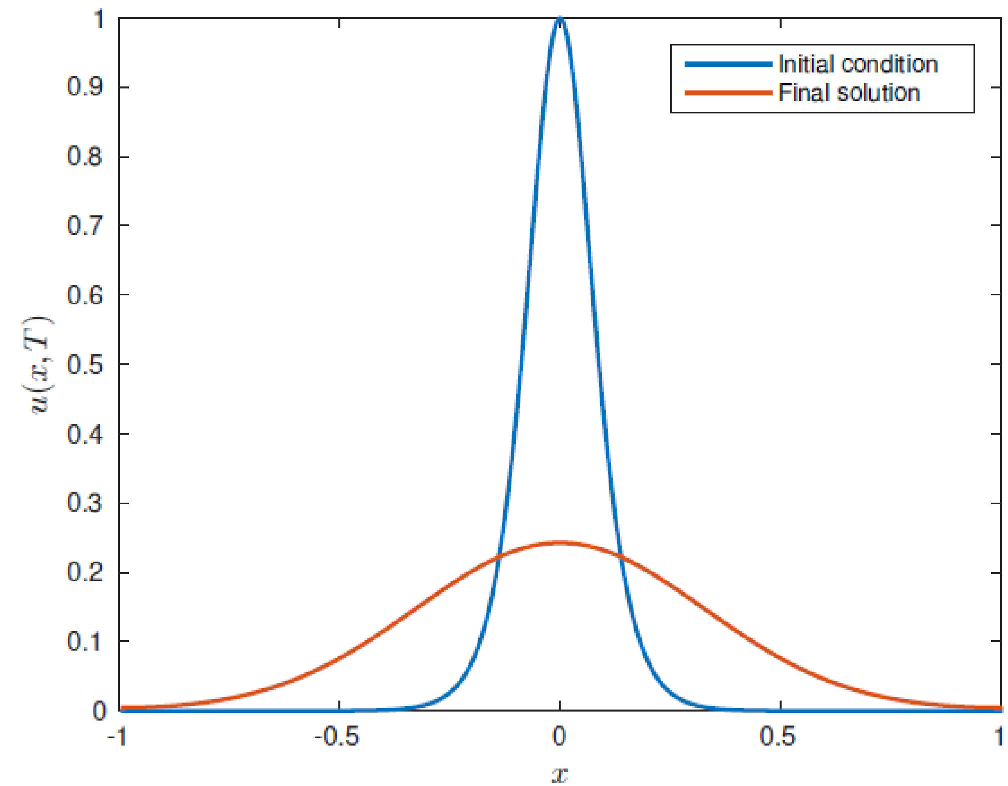


Figure 6. *Spectral solution to the linear heat equation* (4.1) *at* $t = 5.0$, *with* $\nu = 10^{-2}$ *and the initial condition is* $u_0(x) = \mathrm{sech}^2(10\,x)$.

Example

```
l = 1.0;           % half-length of the domain
N = 256;           % number of Fourier modes
dx = 2*l/N;        % distance between two collocation points
x = (1-N/2:N/2)*dx; % physical space discretization
nu = 0.01;         % diffusion parameter
T = 5.0;           % time where we compute the solution
dk = pi/l;         % discretization step in Fourier space
k = [0:N/2 1-N/2:-1]*dk; % vector of wavenumbers
k2 = k.^2;         % almost 2nd derivative in Fourier space

u0 = sech(10.0*x).^2; % initial condition
u0_hat = fft(u0); % Its Fourier transform

% and the solution at final time:
uT = real(ifft(exp(-nu*k2*T).*u0_hat));
```

Comparison

Spectral Method:

Linear combination of continuous functions;

Global approach;

Finite Difference, Flux conservation:

Array of piecewise functions;

Local approach;

+ / -

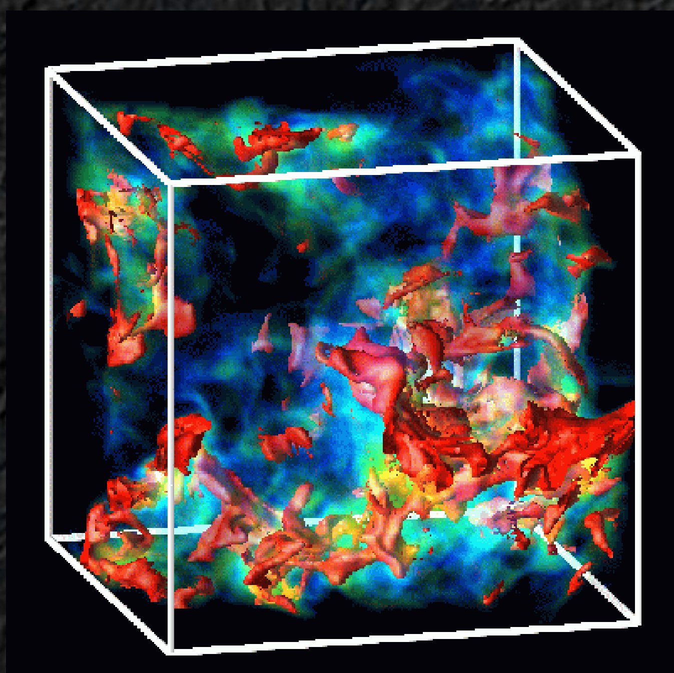
+ (very) fast for smooth solutions
+ Exponential convergence
+ Best for turbulent spectrum

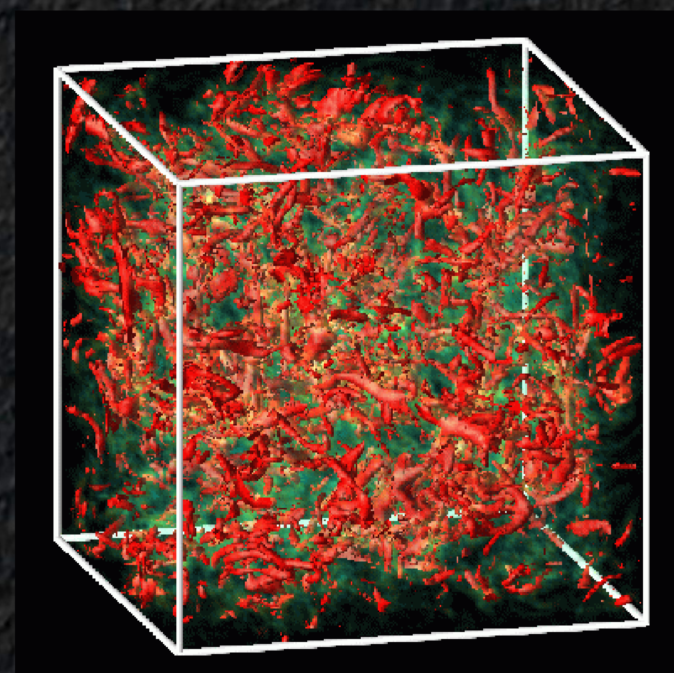
- Shocks
- Inhomogeneities
- Complex Boundaries
- Need for serial reconstruction (integration)

Chaotic flows

Post-processing:

Partial Reconstruction at different length-scales





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